Cryptography

4 – Public-key encryption: RSA

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ALL IS DIGITAL!
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yncréa

Today

Principle

Cryptanalysis

Key generation

In practice

Asymmetric cryptography

Two different keys are used: one for encryption, one for decryption

if knowledge about one gives no information about the other

$\Longrightarrow$ one of them can be made public

Public-key encryption

The encryption key k_e is made public (k_d kept private)

anyone can write to Bob, but only he can read

As implemented by *e.g.* PGP/GPG

Famous "asymmetric" problems

- factorization of large integers

  $\Longrightarrow$ RSA

- discrete logarithm problem (DLP)

  $\Longrightarrow$ Diffie-Hellman, ElGamal, DSA

- DLP over an elliptic curve

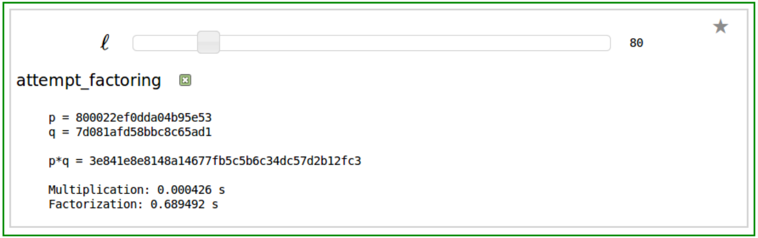
  $\Longrightarrow$ elliptic curve cryptography (ECC): ECDH, ECDSA, ...

- shortest vector problem

  $\Longrightarrow$ lattice-based cryptography ...

For two ℓ-bit factors

Factorization is asymptotically *much* slower than multiplication



Try it for yourself

Modular arithmetic

Recall (?)

Definition

We say that $a \underset{n}{\equiv} b$ when n divides $b - a$, *i.e.* $b = a + kn$ for some integer k

i.e. a and b are equal, up to ("modulo") a multiple of n

Remarks:

- $a \underset{n}{\equiv} b$ if and only if $a \,\%\, n = b \,\%\, n$
- If $a \underset{n}{\equiv} b$ and $c \underset{n}{\equiv} d$, then $(a + c) \underset{n}{\equiv} (b + d)$ and $(ac) \underset{n}{\equiv} (bd)$

Rivest-Shamir-Adleman (1977)

Fix some (large) integer n.

$\mathcal{M} = \mathcal{C} = \mathbb{Z}/n\mathbb{Z}$, identified with $[\![0, n[\![$

$$\begin{cases} E(e, m) :\underset{n}{\equiv} m^e \\ D(d,\ c) :\underset{n}{\equiv} c^d \end{cases}$$

based on **modular exponentiation**

Easy enough!

ℓ 20

```
n = cdf45
m = 34d78
e = b8ee6
m**e mod n = 16b88
```

Or is it? (try a larger ℓ)

Modular exponentiation

Naive algorithm to compute $m^e \,\%\, n$:

$$
\begin{aligned}
& r = 1 \\
& \text{for } i \text{ in } [\![1, e]\!]: \\
& \quad r = r * m \\
& \text{return } r \,\%\, n
\end{aligned}
$$

Problems:

- intermediate result r gets LARGE
- takes e iterations

Modular exponentiation (again)

Better algorithm to compute $m^e \,\%\, n$:

$$
\begin{aligned}
&r = 1\\
&\text{for } i \text{ in } [\![1, e]\!]:\\
&\quad r = (r * m) \,\%\, n\\
&\text{return } r
\end{aligned}
$$

But:

- still takes e modular multiplications ...

Fast exponentiation, v.1 (R to L)

Write $e = b_\ell \cdots b_0$ in base 2, so that $m^e \underset{n}{\equiv} m^{b_0}(m^2)^{b_1}(m^4)^{b_2} \cdots (m^{2^\ell})^{b_\ell}$.

$$
\begin{aligned}
&r = 1, q = m \\
&\text{for } i \text{ in } [\![0, \ell]\!]: \\
&\quad \text{if } b_i = 1: \\
&\quad\quad r = (r * q) \,\%\, n \\
&\quad q = q^2 \,\%\, n \\
&\text{return } r
\end{aligned}
$$

at most $2(\ell + 1)$ modular multiplications!

Example (v.1)

Let's compute 33^{29} modulo 227.

With $m = 33$, $n = 227$ and $e = 29 = 11101$:

i		0	1	2	3	4
b		1	0	1	1	1
q		33	181	73	108	87
r	1	33	33	139	30	113

so $33^{29} \underset{227}{\equiv} 113$ (indeed).

Fast exponentiation, v.2 (L to R)

Can get rid of the running variable q by writing

$$m^e \underset{n}{\equiv} \left(\cdots\left(\left(m^{b_\ell}\right)^2 m^{b_{\ell-1}}\right)^2 m^{b_{\ell-2}} \cdots m^{b_1}\right)^2 m^{b_0}.$$

$r = 1$
for i in $[\![0, \ell]\!]$:
    $r = r^2 \,\%\, n$
    if $b_{\ell-i} = 1$:
        $r = (r * m) \,\%\, n$
return r

In both cases: running time in $\mathcal{O}(\log_2 e)$

Example (v.2)

With the same values as before:

i		0	1	2	3	4
b		1	1	1	0	1
r	1	33	71	189	82	113

which is coherent with previous results (but uses half the memory).

Ok: that's fast

ℓ 33

```
n = cda20ec6
m = 89d3c768
e = a2f8ff3c
m**e mod n = 86df516
```

Indeed!

The RSA cipher (again)

$$\begin{cases} E(e, m) \underset{n}{\equiv} m^e \\ D(d, c) \underset{n}{\equiv} c^{\,d} \end{cases}$$

Correct decryption:

Why should there exist such exponents such that

$$m^{de} \underset{n}{\equiv} m \qquad \forall_m \quad ??$$

Chinese Remainder Theorem

If n can be written as a product of coprime factors

$$n = n_1 \cdots n_k,$$

then there is an *isomorphism of rings*

$$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_k\mathbb{Z}.$$

- $(\rightarrow)$ take remainders
- $(\leftarrow)$ use Bézout's relation

Example

$$\mathbb{Z}/12\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$$

	0	1	2	3
0	0	9	6	3
1	4	1	10	7
2	8	5	2	11

Euler's φ function

Consider the number $\varphi(n)$ of integers in $[\![1, n]\!]$ that are coprime with n.

Theorem (Fermat)

For all x coprime with n,

$$x^{\varphi(n)} \underset{n}{\equiv} 1.$$

i.e., modular exponents work modulo $\varphi(n)$: $x^a \underset{n}{\equiv} x^b$ when $a \underset{\varphi(n)}{\equiv} b$.

Almost there

Special case: suppose $n = p_1 \cdots p_k$ is a product of distinct prime factors, so that

$$\varphi(n) = (p_1 - 1) \cdots (p_k - 1).$$

Corollary

In this case, if $f \underset{\varphi(n)}{\equiv} 1$ *then* $x^f \underset{n}{\equiv} x \quad \forall_x$.

Hence: it is sufficient to ask that the RSA exponents satisfy

$$de \underset{\varphi(n)}{\equiv} 1.$$

A small (thus very insecure) working example

```
n   = 74989
phi = 69600
e   = 52027
d   = 10963

d*e mod phi = 1

message:    60211
encryption: 13247
decryption: 60211
```

Try here

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Security of RSA

Public: n, e, c.

The attacker would like to recover m.

- Brute force on m: search for x such that

$$x^e \underset{n}{\equiv} c.$$

$\Longrightarrow$ Impractical if n large

- Better: try to recover the decryption exponent d, then decrypt m like Bob

$$m \underset{n}{\equiv} \sqrt[e]{c} \underset{n}{\equiv} c^d.$$

Trivial is $\varphi(n)$ is known

Given e and $\varphi(n)$, the extended Euclidean algorithm easily solves

$$de \underset{\varphi(n)}{\equiv} 1.$$

But: computing $\varphi(n)$ from n is (assumed to be) hard.

Best known algorithm: **factor** $n = p_1 \cdots p_k$ and use

$$\varphi(n) = (p_1 - 1) \cdots (p_k - 1).$$

Factoring vs. splitting

Factoring n: finding the complete list of prime factors $(p_1, \ldots, p_k)$ for which

$$n = p_1 \cdots p_k.$$

Splitting n: finding *one* prime factor p of n.

Essentially all known factorization algorithms are of the form

$$\begin{array}{l} \text{factors} = [\ ] \\ \text{while } n > 1\text{:} \\ \quad p = \text{split}(n) \\ \quad \text{factors} \mathrel{+}= [\ p\ ] \\ \quad n = n \ /\!/ \ p \end{array}$$

Trial division

The simplest splitting algorithm:

$$
\begin{aligned}
&p = 2\\
&\text{while } p \leq \sqrt{n}\text{:}\\
&\quad \text{if } n \,\%\, p = 0 \text{ return } p\\
&\quad p \mathrel{+}= 1\\
&\text{return } n
\end{aligned}
$$

Quickly finds small ($\leq 2^{64}$) prime factors

$\implies$ smallest prime factor should be as large as possible

$p_i \approx \sqrt[k]{n} \implies$ take $k = 2$! (why not $k = 1$?)

Other factorization algorithms

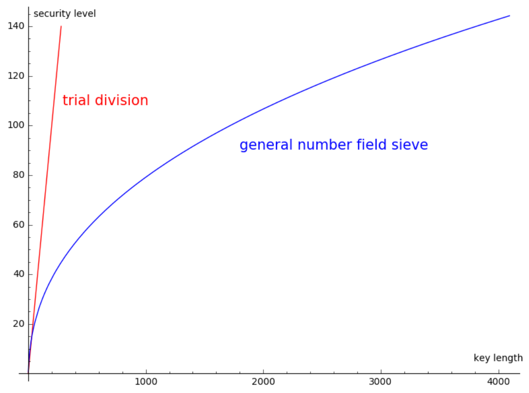
There is a very large litterature devoted to the subject of integer factorization.

As of 2019, the best general purpose algorithm is the *General Number Field Sieve* (GNFS) that factors an ℓ-bit integer in

$$\approx 5.5^{\,\ell^{1/3}(\ln \ell)^{2/3}} \text{ time.}$$

Public factorization record: RSA-728 (2009)

Consequence on key length



According to RSA Security, Inc.

Symmetric key size	Equivalent RSA key size
80	1024
112	2028
128	3072
256	15360

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Recovering the decryption key should be hard for the attacker...

...but easy for Alice and Bob!

Ok since they are free to choose the prime factors of n.

Key generation: produces a RSA triple (n, d, e)

Prime factors

To generate an ℓ-bit RSA modulus n:

- generate two random $\ell/2$-bit prime numbers p and q
- set $n := p \cdot q$

To generate a random prime number:

- generate random integers until you get a prime!

(there are some very fast primality tests)

Note: density of prime numbers around x is $\approx \frac{1}{\ln x}$

RSA exponents

- Knowing p and q, compute $\varphi(n) = (p-1)(q-1)$
- Pick e coprime with $\varphi(n)$ (doesn't even need to be chosen randomly)
- Compute d such that

$$de \underset{\varphi(n)}{\equiv} 1$$

  using the extended Euclidean algorithm (XGCD)

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Real-world RSA

The *plain RSA* described above has all sorts of problems:

- malleability: $E(e, m_1) \cdot E(e, m_2) = E(e, m_1 \cdot m_2)$
- lack of randomness
- fixed size of plaintext
- ...

In practice, a suitable padding scheme needs to be used.

$\implies$ use a library!